On Hilbert's epsilon operator in Form Sequence

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Abstract

This paper examines the role of Hilbert's ϵ-operator in Chomsky's (2019, 2020) Form Sequence theory for coordination. Inspired by Chomsky's idea, I develop an implementation of the theory within current Minimalism. I propose that Form Sequence is a two-thread procedure, with its ϵ-thread being in the discourse and its coordination thread, in syntax. With this separation, we can retain the purely hierarchical nature of syntactic representation and broaden the application of the ϵ-method. For example, it can also generate the salience ranking in the Konstanz School's semantic analysis of definite NPs. Furthermore, I argue for the domain-general nature of the ϵ-method, as exemplified by its use in multitasking prioritization, and identify it as a third-factor strategy à la Chomsky (2005).

Keywords: Minimalism, Hilbert's epsilon operator, Form Sequence, Pair Merge, coordination, third factor

1 Introduction

Hilbert's epsilon operator (henceforth ϵ-operator), named after the German mathematician David Hilbert, is a fundamental symbol from *Grundlagen der Mathematik* (Hilbert & Bernays 1939). An ϵ-operator, which is more exactly a subnector (Curry 1963:32–33), forms a term out of a formula, as in (1).

(1) $\epsilon x.\,\mathrm{F}(x)$

Here, x is an individual variable, F is a first-order predicate, and the entire string is an ϵ-term, which roughly means "an individual x such that F is true for x." Thus, if F is *apple*, then the denotation of (1) is a particular apple. In Hilbert's original conception, ϵ-terms are nondeterministic, so there is no way to know precisely which individual (e.g., which apple) is picked. In this sense, the ϵ-term in (1) intuitively corresponds to the indefinite description "an F."

In this paper, I examine the role of the ϵ-operator in Chomsky's new theory of Form Sequence, which he introduced in two recent lectures—his 2019 UCLA lecture and his 2020 Linguistics Society of Japan lecture—as a latest addition to the Minimalist Program. Chomsky's idea is that the ϵ-operator could be part of an extended theory of Pair Merge (Chomsky 2000), which can help us derive "unbounded unstructured coordination." The phenomenon is exemplified in (2).

(2) a. I met someone [young, happy, eager to go to college, tired of wasting his time, . . .]

b. The guy is [young, tall, happy, young, eager to go to Harvard, . . .]

c. [John, Bill, Tom, the young man, . . .] [read the book, walked to the store, . . .] (Chomsky 2019)

Note that (2b) has two occurrences of *young*, which shows that coordinated items in a sequence may repeat. The bracketed coordinate phrases in (2) are "unbounded" in that they can go on and on without upper bound. And since within each coordinate phrase no conjunct is in the scope of any other conjunct, the coordinations are all "unstructured"—in the technical sense that there is no asymmetrical c-command relation between the conjuncts. To generate sequences of conjuncts like the above, Chomsky (2019) resorts to the Minimalist operation for adjunction—Pair Merge—and lets each component S_i of a coordination sequence S pair-merge with a link element L_i (and all links in a coordination are in fact identified as one and the same). He then places all the $\langle S_i, L_i \rangle$ pairs in a sequence, as in (3a), the first slot of which is occupied by a conjunction. Chomsky (2020) presents the same idea in a slightly different form, as in (3b).

(3) a. $\langle \text{CONJ}, \langle S_1, L_1 \rangle, \ldots, \langle S_n, L_n \rangle \rangle$ (Chomsky 2019)

b. $\langle (\&), X_1, \ldots, X_n \rangle$ (Chomsky 2020)

(& is an optional conjunction and each X_i is a conjunct)

In Chomsky (2020), the operation that gives rise to such a coordination sequence is named Form Sequence (the term is not yet coined in Chomsky 2019 but the idea is already there). Importantly, in Chomsky's conception the formation of such a sequence involves the choice of a particular interconjunct ordering out of a set of alternatives, and this is where Hilbert's ϵ-operator comes into play. The following quote is from Chomsky (2019):

> [I]n order to generate these objects, you generate a finite set, and then you form from that set a sequence. It could be any sequence of elements, and there's in fact infinitely many possible sequences. You pick one out of those, and that sequence, call it S, is the thing that you are then going to merge into the construction. This operation of picking a particular element out of the set of sequences is David Hilbert's epsilon operator, which picks a single thing out of a set. It was part of his work on the foundations of mathematics—[a] basic operation. It's a straightforward operation, but it does have the property of being indeterminate.

The Form Sequence theory is still in its preliminary stages, and Chomsky has only mentioned the ϵ-operator part of it in passing—indeed, only in the above quote. Nevertheless, it is clear from the quote that the ϵ-operator plays a key role in the definition of Form Sequence. Furthermore, it is clear from Chomsky's lectures that Form Sequence qua an upgrade of Pair Merge has more theoretical significance in the Minimalist Program beyond coordination. Thus, Chomsky (2020) suggests that the simple phrases we normally see—such as *John saw Bill* and *John ran*—are "just limiting cases of sequences" and that "wherever there is an XP, there would be a sequence." Given the potentially fundamental place of sequence in the Minimalist Program, I believe it is worthwhile to carefully examine this data structure and its generation within current syntactic theory.

Against this background, the main purpose of this paper is to build on Chomsky's idea and explore how exactly Hilbert's ϵ-operator can help us generate multimembered sequences. Crucially, while my point of departure is the programmatic proposal in Chomsky's 2019 and 2020 lectures, the main body of this paper and especially its technical details are my own development. Due to the limited scope of a single paper, I must leave aside some significant questions about Form Sequence, especially questions about its motivation, such as "Why is unbounded unstructured coordination a problem for the theory of grammar?" and "Why should we adopt Form Sequence instead of alternative operations?" Such questions are valid but noncentral to my immediate purpose. In what follows, I simply follow Chomsky in assuming that sequence is a necessary data structure in generative syntax and that some extension of Pair Merge is useful in the derivation of coordinate phrases. The contribution of the present paper is threefold. First, it highlights and tentatively answers some theoretical questions in relation to Form Sequence. Second, it examines the use of the ϵ-operator hinted by Chomsky in detail. Third, it links the ϵ-method in Form Sequence to the "third factor" perspective in Chomsky (2005).

The remainder of the paper is organized as follows. In Section 2, I give a more formal introduction of the ϵ-operator. In Section 3, I first review the connection between Pair Merge and Form Sequence and then list some questions about the latter and the role of the ϵ-operator therein. In Section 4, I address those questions and work out an implementation of Form Sequence within current Minimalism. In Section 5, I further show that the ϵ-method in Form Sequence has more applicability beyond syntax and even beyond the domain of language, which leads me to view it as a third-factor strategy in the sense of Chomsky (2005). In Section 6, I summarize the main results of the paper.

2 The ϵ-operator

Before examining the functionality of the ϵ-operator in Form Sequence, I first introduce this mathematical logical symbol in a more formal setting. The content in this section is mainly based on Avigad & Zach (2020), Chatzikyriakidis, Pasquali & Retoré (2017), Leisenring (1969), and Slater's entry in the *Internet Encyclopedia of Philosophy*. There is controversy among logicians about certain aspects of the ϵ-operator (e.g., about its semantic model), but the general picture presented below suffices for current purposes.

Partly inspired by Russell's iota operator for definite descriptions (Whitehead & Russell 1910), Hilbert proposed two generic element symbols in the 1920s—first τ and then ϵ. See (4) for an unfolding of the three symbols.

(4) a. $\iota x.\, \mathrm{F}(x)$: the unique x that satisfies F

b. $\tau x.\, \mathrm{F}(x)$: an x that satisfies F when every individual does so

c. $\epsilon x.\, \mathrm{F}(x)$: an x that satisfies F when some individual does so

Unlike ι, which basically says *the*, both τ and ϵ return generic elements, and in principle there is no way to know exactly which individual is chosen. For this reason, Hilbert's two operators are said to be nondeterministic (or indeterminate). Moreover, τ and ϵ are closely related to the two quantifiers $\forall$ and $\exists$ in predicate logic. Indeed, (4b) and (4c) are respectively a universal and an existential generic object with regard to F (Chatzikyriakidis et al. 2017), as in (5).

(5) a. $\mathrm{F}(\tau x.\, \mathrm{F}(x)) \equiv \forall x.\, \mathrm{F}(x)$

b. $\mathrm{F}(\epsilon x.\, \mathrm{F}(x)) \equiv \exists x.\, \mathrm{F}(x)$

While τ and ϵ had started their lives as different symbols, they are in fact mutually definable (see, e.g., Retoré 2014 and Abrusci 2017), so in the end Hilbert only kept ϵ.

Hilbert's original purpose with the ϵ-operator was to find a consistent and complete axiom set for mathematics, first and foremost for arithmetic. The ϵ-operator was convenient for this purpose in that it eliminated the two quantifiers and could also replace the Axiom of Choice (see, e.g., Bernays 1991/1958). Hilbert's program eventually failed due to Gödel's (1931) incompleteness theorems, according to which there is no consistent axiomatic system for arithmetic, and no system can prove its own consistency. Nevertheless, Hilbert's endeavor left us with a number of valuable results, including ϵ.

The ϵ-operator, as a logical symbol, should be equipped with an ambient syntax and a corresponding semantics. Its syntax is known as the ϵ-calculus, which is a minimal extension of predicate calculus, with ϵ being the only new symbol. The ϵ-calculus defines an ϵ-term for each and every predicate, and for each ϵ-term there is a corresponding axiom known as Axiom ϵ.

(6) Axiom ϵ: $\mathrm{F}(t) \rightarrow \mathrm{F}(\epsilon x.\, \mathrm{F}(x))$

"If any term t has the property F at all, then $\epsilon x.\, \mathrm{F}(x)$ has it."

What this axiom says is essentially that for any nonempty subset of the domain of discourse, we can choose a representative element from it, but that is basically the Axiom of Choice. Indeed, Hilbert's ϵ-operator is also known as the choice operator.

Hilbert did not give ϵ any semantics at the time of its proposal but merely used it as a syntactic tool to facilitate proof construction. Asser (1957) interpreted ϵ as a choice function in the following model:

(7) $\mathcal{M} := \langle \mathbf{J}, \mathbf{I} \rangle$ (based on Asser 1957:33–34)

Here, $\mathcal{M}$ is the model, $\mathbf{J}$ is its domain, and $\mathbf{I}$ is its constant-interpreting function. Asser sets $\mathbf{I}(\epsilon)$ to be a choice function Φ, which chooses an arbitrary element from each subset of $\mathbf{J}$. Asser also took into consideration the empty set—namely, the case where the *if*-clause in (6) is false. He suggested two possible solutions, one with a total choice function and the other with a partial one. On the total function solution, $\Phi(\emptyset)$ returns an arbitrary element ξ_0 of $\mathbf{J}$—namely, an arbitrarily chosen individual in the whole world—and on the partial function solution it is undefined. Thus, assuming $[\![F]\!] = A$, we have

(8) $$[\![\epsilon x.\, \mathrm{F}(x)]\!] = \Phi([\![\mathrm{F}]\!]) = \Phi(A \subseteq \mathbf{J}) = \begin{cases} a \in A, & \text{if } A \neq \emptyset \\ \xi_0 \in \mathbf{J} \text{ or undefined}, & \text{if } A = \emptyset \end{cases}$$

As Leisenring (1969) points out, the total function solution suits Hilbert's original purpose better. This is also the sentiment in some later works on the philosophy of language. For instance, Slater (2017:278) explicates that if there is no such x that satisfies $\mathrm{F}(x)$, then the denotation of $\epsilon x.\, \mathrm{F}(x)$ "is a fiction, which means it is simply a pragmatically chosen individual in the whole world at large."

In the above, my introduction of the ϵ-operator has been limited to the first order, where the ϵ-bound variable is of the individual type. But ϵ can obviously also bind higher-order objects. See Ackermann (1925) and Hilbert & Bernays (1939:Supplement IV.A) for second-order ϵ-terms of the form $\epsilon f.\, \mathrm{A}(f)$, where f is a function variable. As I will show in Section 4.2, the ϵ-term in Form Sequence is also of the second order, with the ϵ-bound variable being of the ordered set type. Furthermore, in the Konstanz School's use of ϵ in formal semantics to be reviewed in Section 5.1, the choice function itself becomes a matter of choice too, for which purpose context-indexed ϵ-terms of the form $\epsilon_i x.\, \mathrm{F}(x)$ are used, where i stands for the particular context in which the ϵ-choice is made. See Mints & Sarenac (2003) and Leiß (2017) for the semantics of indexed ϵ-calculus.

In sum, Hilbert's ϵ-operator had originally been proposed as part of a program to completely axiomatize mathematics. Despite the failure of Hilbert's program, his ϵ-tool survived till this day and has been influential in several disciplines, including linguistics. The formal system ϵ lives in is the ϵ-calculus, and it is semantically interpreted as a choice function. In addition, the ϵ-tool can be flexibly applied to objects of various types.

3 From Pair Merge to Form Sequence

As mentioned in Section 1, Chomsky's use of the ϵ-operator is part of an extended theory of Pair Merge. In current Minimalist Syntax (since Chomsky 2000), Pair Merge is used to generate adjunction structures. It takes two syntactic objects α and β as input and returns an ordered pair $\langle \alpha, \beta \rangle$ as output, where α is the adjunct and β is the host. Pair Merge is different from Set Merge in that set elements are unordered while pair components are ordered, as in (9).

(9) a. Set Merge$(\alpha, \beta) = \{\alpha, \beta\} = \{\beta, \alpha\}$

b. Pair Merge$(\alpha, \beta) = \langle \alpha, \beta \rangle \neq \langle \beta, \alpha \rangle$

Chomsky (2004:117–118) further likens adjunction to higher-dimensional structure building, suggesting that "we might intuitively think of α as attached to β on a separate plane, with β retaining all its properties on the 'primary plane,' the simple structure." This feature of Pair Merge is inherited in its Form Sequence extension. In Chomsky's (2019) words: "The unbounded unstructured cases show [that] there are unboundedly many dimensions to what's going on up there [in the mind]. [It's] not two-dimensional like a blackboard. You can add any number of adjuncts at any point." I illustrate this dimension-expanding capacity of Pair Merge in (10), where n adjuncts are attached to a single host.

(10) Pair Merge$(\alpha_1, \beta) = \langle \alpha_1, \beta \rangle$

Pair Merge$(\alpha_2, \beta) = \langle \alpha_2, \beta \rangle$

...

Pair Merge$(\alpha_n, \beta) = \langle \alpha_n, \beta \rangle$

Chomsky (2019) assumes that in the derivation of unbounded unstructured coordination, each conjunct-link unit—namely, each $\langle \mathrm{S}_i, \mathrm{L}_i \rangle$ in (3a)—is a pair-merged object. In addition, all links in the same coordination are actually one and the same, because "[we] are attaching everything to the same point."

Both Set Merge and Pair Merge are simple, basic operations in the Minimalist Program. By comparison, Form Sequence is not that simple. The quote from Chomsky (2019) in Section 1 specifies three steps for it: (i) generate a finite set; (ii) form from that set a set of sequences; and (iii) choose a particular sequence. I use the generation of *young, tall, and happy* to illustrate this, as in (11).

(11) a. $\{\text{young}, \text{tall}, \text{happy}\}$ (a set of conjuncts)

b. $\{\langle \&, \langle \text{young}, L \rangle, \langle \text{tall}, L \rangle, \langle \text{happy}, L \rangle \rangle,$ (a set of sequences)
$\langle \&, \langle \text{tall}, L \rangle, \langle \text{young}, L \rangle, \langle \text{happy}, L \rangle \rangle,$
$\langle \&, \langle \text{happy}, L \rangle, \langle \text{tall}, L \rangle, \langle \text{young}, L \rangle \rangle, \dots\}$

c. $\langle \&, \langle \text{young}, L \rangle, \langle \text{tall}, L \rangle, \langle \text{happy}, L \rangle \rangle$ (a particular sequence)

There are quite a few puzzles in the sketch of Form Sequence in Chomsky (2019, 2020). The idea is appealing, but its implementation requires much more details to be filled in. I list some questions that I deem crucial in the remainder of this section and present a viable implementation based on my answers to them in Section 4.

The dissection of Form Sequence in (11) shows several seemingly anti-Minimalist characters. First, the set in (11a) is multimembered and cannot be formed by Set Merge, which is binary by definition. Second, there is a conjunction in the sequences in (11b–c) but none in the initial set in (11a), which violates the Inclusiveness Condition (Chomsky 1995:225) as quoted below:

> Another natural condition is that outputs consist of nothing beyond properties of items of the lexicon (lexical features)—in other words, that the interface levels consist of nothing more than arrangements of lexical features. To the extent that this is true. the language meets a condition of inclusiveness.

Assuming the conjunction in (11b–c) ultimately comes from the lexicon, and that (11a) precedes (11b) in the operation Form Sequence, we must ask at what stage it joins in. Third, the set in (11b) contains "infinitely many possible sequences" in Chomsky's conception, but syntactic derivation is finitely defined. Thus, it is puzzling how this critical step can ever be made part of a derivation.

There is a further puzzle concerning the connection between Pair Merge and Form Sequence. Pair Merge generates objects of the form $\langle\alpha, \beta\rangle$. However, in Chomsky's (2020) illustration of Form Sequence, as in (12), no conjunct-link unit is in this form.

(12) a. John arrived and met Bill.

b. $\{\mathrm{C}, \{\mathrm{John}_3, \{\mathrm{INFL}, \underline{\langle \&, \{_1 \{_2\ v, \{\text{arrive John}_1\}\}\}, \{_3\ \mathrm{John}_2, \{v^*, \{\text{meet B}\}\}\}\rangle}\}\}\}$

Chomsky (2019) assumes the link elements in Form Sequence are v and n. This means that each conjunct and v/n together form a $\langle \mathrm{S}_i, \mathrm{L}_i\rangle$ pair. However, this is clearly not the case in (12b), where the underlined coordination sequence contains no conjunct-link *pairs*. The $\langle \mathrm{S}_i, \mathrm{L}_i\rangle$ slots in (3a) are filled in by ordinary sets instead. The structure in (12b) is more in line with the alternative sequence format in (3 b)—namely, $\langle(\&), \mathrm{X}_1, \ldots, \mathrm{X}_n\rangle$—but since the conjunct-link pairs are what connect Pair Merge and Form Sequence, their absence from the alternative format undermines this conceptual connection in a fundamental way.

4 Form Sequence with ϵ

4.1 The relevance of the discourse

As mentioned above, Form Sequence comprises three steps: conjunct set generation, sequence space formation, and sequence choosing. The first two steps are anti-Minimalist in that they violate established syntactic principles: the binarity of Set Merge and the inclusiveness and finiteness of syntactic derivation. The third step is just the ϵ-choice, but a closer look reveals that it does not look very "syntactic" either—for it neither builds up syntactic objects nor manipulates them, unlike familiar syntactic rules (e.g., Merge, Agree).

As far as I am concerned, the effect of Form Sequence is more like that of a "backstage" process rather than a process in Narrow Syntax proper. That is, Form Sequence—or at least the ϵ-part of it—serves to generate certain extra information that may be consulted by the syntactic interfaces but is not part of the narrow syntactic representation. Abusing the term "discourse" a bit, I propose that this extra sequential information is a type of discourse information, on a par with referents, presuppositions, attitudes, and so on.

To illustrate the relevance of discourse information for syntax, consider the simple sentence below.

(13) She saw him.

When a speaker utters (13) out of the blue, what they assume is a lot more than what they say. To begin with, there must be two particular persons—one female and the other male—in their mental discourse. Besides, they must also have a particular eventuality in mind, which is additionally anchored to a particular temporal interval. None of these is explicitly conveyed, and hence none of them is derived in Narrow Syntax. Yet such discourse information can affect syntactic derivation—more exactly its initializing stage—in the background. It is the gender/number information that guides the selection of the appropriate pronouns (or feature bundles thereof) from the lexicon and the temporal information that guides the selection of the appropriate tense feature. In the generative syntactic literature, the selection of lexical items is rarely examined in detail, but it evidently relies on information in the speech context—namely, the discourse. The above-mentioned gender/tense features are but two common examples.

Another domain where syntactic derivation is clearly affected by discourse information is that of the various left-periphery phenomena, such as topicalization and focalization. In some languages, such as Hungarian, topicalized or focalized constituents occupy dedicated positions in the sentence, as in (14).

(14) [$_{\text{TopP}}$ *Marit*$_i$] [$_{\text{FocP}}$ *János*$_j$] *kérte fel* t_i t_j. [Hungarian]
Mary.ACC John.NOM asked up

'As for Mary, it was John who invited her (for a dance).'

(adapted from É. Kiss 2002:3)

The R-expressions 'Mary' and 'John' are clearly not specified as [+TOPIC] or [+FOCUS] in the lexicon, so their being sensitive to movement driven by such features can only be due to discourse interference—probably via what Chomsky (1995:231) terms "optional features," as defined in the following quote:

> The collection of formal features of the lexical item LI I will call *FF(LI)*, a subcomplex of LI. . . . Some of the features of FF(LI) are *intrinsic* to it, either listed explicitly in the lexical entry or strictly determined by properties so listed. Others are *optional*, added as LI enters the numeration. . . . In the case of *airplane*, the intrinsic properties include the categorial feature [nominal], the person feature [3 person], and the gender feature [−human]. Its optional properties include the noncategorial features of number and Case.

What topic and focus exactly are in the discourse and how exactly they are generated in the speaker's mind are complex issues, but that complexity is irrelevant here—what matters is that at the beginning of the derivation, the lexical items 'Mary' and 'John' are already associated with the two discourse features, which in turn guides syntactic operations like Move (i.e., Internal Merge).

What I propose here is that the sequential information needed in the proper linearization and interpretation of coordinate phrases also belongs to the discourse, though unlike in the two cases above, the sequential information does not enter Narrow Syntax at all (i.e., there are no relevant formal features) but stays in the discourse throughout. It can nevertheless be consulted by the PF/LF interfaces. I will elaborate on this in Section 4.4.

4.2 The sequence space as a free monoid

In the foregoing, I proposed that Form Sequence may not be a totally narrow-syntactic operation but may involve certain discourse processes instead. On this conception, the infinite set of sequences (i.e., the sequence space) in (11b) is no longer a mystery, since infinite sets may well exist in the discourse, as evidenced by expressions like *everything* and *all natural numbers*. But if the sequence space is in the discourse, then the ϵ-choice made from it necessarily also belongs to the discourse. That is indeed what I assume, as mentioned at the beginning of the previous section.

In this section, I present the above conception more formally. Specifically, the set of all possible sequences generated from a given set A (assuming these are all finite) is just the free monoid A^* on A, whose identity element is the empty sequence $\langle\,\rangle$ and whose monoid operation is concatenation (notated by ++ below). See (15) for the definitions of monoid and free monoid.

(15) a. A monoid $\langle M, \cdot, e\rangle$ is a set M equipped with an associative binary operation $\cdot$ and an identity element e such that $\forall m \in M$, $e \cdot m = m \cdot e = m$.

b. The free monoid on a set has as elements all finite sequences generated from zero or more elements of that set by concatenation.

Thus, the free monoid on $\{a, b\}$ is $\{\langle\,\rangle, \langle a\rangle, \langle b\rangle, \langle a, b\rangle, \langle a, a\rangle, \langle a, a, b\rangle, \dots\}$, where singleton entries like $\langle a\rangle$ are generated by vacuous concatenations like $\langle a\rangle$++$\langle\,\rangle$. What the ϵ-operator does in the Form Sequence procedure is pick an item out of the free monoid A^* on an initial conjunct set A. I give this ϵ-term in (16).

(16) $\epsilon X.\, \mathrm{seq}_A(X)$

This ϵ-term chooses a particular sequence from the free monoid on A. I use the uppercase X as the ϵ-bound variable here because, strictly speaking, this is not a first-order ϵ-term, for each sequence is essentially an ordered set. Regardless of that, the ϵ-operator works in the same way as in the first-order scenario.

While the above difference in variable type is less significant, there does exist a more significant difference between the ϵ-term in (16) and Hilbert's original ϵ in mathematics. Recall from Section 2 that in Hilbert's original conception, the ϵ-choice is nondeterministic. That is, a term like $\epsilon x.\, F(x)$ can pick out any member of $[\![F]\!]$ without preference. By contrast, in the case of (16), some members of A^* are in practice never chosen under normal circumstances. These include, among others, sequences with many random repetitions and sequences with fewer components than members of the initial set. Thus, for the initial set $B = \{\text{young}, \text{tall}, \text{happy}\}$, the following sequences are examples of bad candidates (I temporarily ignore the link element and the conjunction):

(17) a. $\langle\text{young}, \text{tall}, \text{young}, \text{young}, \text{tall}, \text{happy}, \text{happy}\rangle$

b. $\langle\text{young}\rangle$

c. $\langle\,\rangle$

In fact, under normally circumstances, the viable candidates of $\epsilon X.\, \mathrm{seq}_B(X)$ are just the following six (i.e., the simple permutations of the initial set):

(18) a. $\langle \text{young}, \text{tall}, \text{happy} \rangle$

b. $\langle \text{young}, \text{happy}, \text{tall} \rangle$

c. $\langle \text{tall}, \text{young}, \text{happy} \rangle$

d. $\langle \text{tall}, \text{happy}, \text{young} \rangle$

e. $\langle \text{happy}, \text{tall}, \text{young} \rangle$

f. $\langle \text{happy}, \text{young}, \text{tall} \rangle$

This is presumably a pragmatic constraint imposed by, for instance, Grice's Maxims, since nothing strictly prohibits a speaker from saying things like *John is young, tall, young, young, tall, happy, and happy* (e.g., for fun) or from starting with the initial set B in mind but ending up only saying *John is young* (e.g., waiting for the interlocutor to continue). The speaker in principle has full freedom in terms of the ϵ-choice, but pragmatics greatly reduces the sequence space from the entire free monoid on the initial set to just the set of its permutations.

There are apparently also cases where the ϵ-choice is truly not free. Consider the following example taken from Chomsky (2019):

(19) [John and Bill] saw [Tom and Mary] respectively.

In (19), the adverb *respectively* imposes a particular interdependence on the two bracketed coordinate phrases. Similarly, I observe that when an initial set is associated with a conventionalized order, the ϵ-choice usually obeys that order, as in (20).

(20) a. The rainbow colors are [red, orange, yellow, green, blue, indigo, and violet].

b. The twelve months are [January, February, March, April, May, June, July, August, September, October, November, and December].

c. The Hogwarts houses are [Gryffindor, Hufflepuff, Ravenclaw, and Slytherin].

In all these examples, the ϵ-choice is made under the guidance of convention or common knowledge, which is a type of discourse information too.

4.3 Multidimensionality

In the last section, I mainly focused on the role of the ϵ-operator in Form Sequence. But after the ϵ-choice is made, which subsequently becomes part of the discourse information, we still need to derive the actual coordinate phrase in syntax. In this section, I turn to this aspect of Form Sequence and pay particular attention to the link element and associated notions. Recall from Section 3 that both Pair Merge and Form Sequence create higher-dimensional objects. Whatever the exact definition of "higher-dimensional" is in the context of syntactic derivation (it seemingly points to a difference in data structure), such objects clearly have some properties that set-merged objects do not possess. I believe that a closer look at such properties can help us resolve the remaining puzzle from Section 3—namely, why the link element from (3a) is not actually pair-merged with its sister constituent in (12b).

Prior to Chomsky's (2019) proposal, a multidimensional structure was already suggested for coordinate phrases in de Vries (2004, 2005)—as a solution to the following question: "[H]ow can we represent the intuitive symmetry of coordination, and in particular, how can we prevent the first conjunct from c-commanding the second?" (de Vries 2005:92) De Vries's theory involved a "behindance" relation in addition to the standardly defined dominance relation, which was based on his conception that "conjuncts are behind each other in a three-dimensional structure" (ibid.). Accordingly, he also proposed a "b-Merge" operation—namely, Merge by behindance. Apart from some technical details, de Vries's and Chomsky's ideas on coordination are almost the same.

Recall from Section 3 that the conjunction is optional in Chomsky's conception of Form Sequence. Deviating from this position but in line with de Vries

(2005), I take the conjunction (i.e., its abstract, featural representation) to be always present in the underlying syntax of coordination and assume that what is optional is merely its phonological exponence. As I will show below, having a conjunction in the syntactic representation not only facilitates syntax-semantics mapping (for it corresponds to the logical AND/OR) but also helps us untangle the discrepancy between (3a) and (12b).

In Chomsky's conception of Form Sequence, each conjunct is attached from a different dimension to "the same point." I call this point the pivot of a coordination structure. Note that the pivot is arguably not the link element, because it is pair-merged with each conjunct, as in (10), whereas the link is subject to Set Merge within its ambient phrase marker, as in (12b). But if the link is not the pivot, then what is? What we know is that the pivot serves to hold the conjuncts together, so it ought to lie at the intersection of all the dimensions in a coordination. In addition, there can be an unbounded number of conjuncts, so the pivot has flexible arity (i.e., it accepts any number of arguments). Based on these criteria, the logical connectives AND/OR are the obvious candidates, which I notate by the umbrella label Co.

Recall that all links in a coordination are identified as one and the same in Chomsky's conception. I take this to be a well-formedness constraint—presumably an interface condition. Satisfaction of this constraint may be what makes a multidimensional coordination labelable. If the link element is v, then the coordinate phrase's real label is vP rather than CoP—for clarity's sake I will notate such a phrase as CovP. This scenario constitutes a special instance of the XP-YP case in Chomsky's (2013, 2015) Labeling Algorithm, where the label is provided by some shared feature(s).

I make a further distinction between the notions "link" and "host." Compared to the link, the host component is much harder to pin down. In a pair-merged object $\langle \alpha, \beta \rangle$, α is the adjunct and β is the host, and the category of the entire object is the same as that of β (i.e., β "projects" in traditional parlance).

For instance, the category of *young man* is the same as that of *man*. However, there is a crucial difference between this classical scenario of Pair Merge and the multidimensional Pair Merge involved in Form Sequence, and we encounter nontrivial trouble if we simply take the pivot of the multidimensional object (i.e., Co) to be the host, even though coordinate phrases are often conveniently labeled as CoP. The trouble is that while the host in classical Pair Merge (e.g., *man* in *young man*) can be used on its own, Co cannot (i.e., it is syncategorematic); nor are the conjuncts modifiers of Co in any sense. Intuitively, if anything in a coordinate phrase projects at all, it should be the conjuncts' shared category—namely, the link—rather than Co. Therefore, if we reserve the term "host" for the labeling component as in standard Pair Merge theory, then the host of each Co-XP pair should be XP instead of Co. This brings us to the somewhat peculiar conclusion that the multidimensional coordinate phrase is actually multihosted, with as many hosts as its dimensions. That being said, these hosts are still not the same as those in classical Pair Merge, for Co is not a modifier of XP in any sense either, just as XP is not a modifier of Co. Henceforth, I will use $\langle \mathrm{Co}, \mathrm{XP} \rangle$ to notate the Co-XP pair and call XP the host, though this designation is more expository than substantive. See Figure 1 for an illustration of the internal relationships of a multidimensional syntactic object. I use a superscript "L" to indicate an XP-internal element that serves as the link element and use dotted lines to indicate Pair Merge.

Under this view of multidimensional coordination, I prefer viewing the Form Sequence structure in Chomsky (2019), which is repeated in (21), as a high-level declaration—like a general description of what Form Sequence is—rather than an actual syntactic object.

(21) $\langle \mathrm{CONJ}, \langle \mathrm{S}_1, \mathrm{L}_1 \rangle, \ldots, \langle \mathrm{S}_n, \mathrm{L}_n \rangle \rangle$ (=(3a))

Specifically, this notation declares that a link element can be identified for each conjunct. In formal terms, this amounts to defining a function $\lambda \mathrm{S}_i.\, \mathrm{L}_i$ that as-

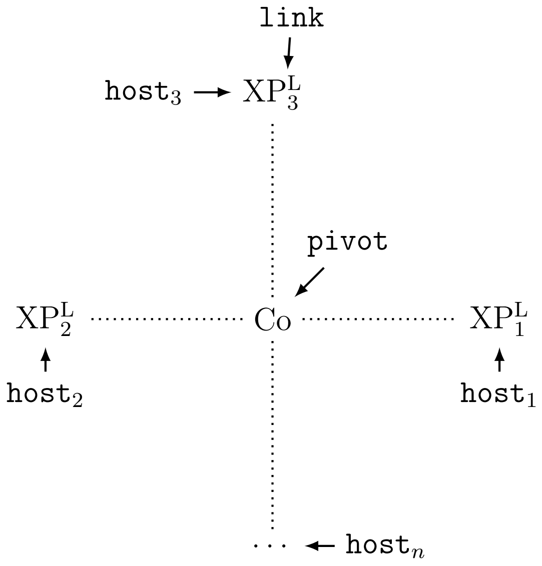


Figure 1: Relationships in a multidimensional CoP

signs to each conjunct term one of its subterms, which in set talk is exactly a set of pairs $\{\langle \mathrm{S}_1, \mathrm{L}_1\rangle, \langle \mathrm{S}_2, \mathrm{L}_2\rangle, \ldots, \langle \mathrm{S}_n, \mathrm{L}_n\rangle\}$. Crucially, the pairs $\langle \mathrm{S}_i, \mathrm{L}_i\rangle$ here are not products of Pair Merge but just a way of metalinguistic presentation. In this sense, the alternative notation $\mathrm{S}_i^{\mathrm{L}_i}$ is less ambiguous.

4.4 Multilayered derivation

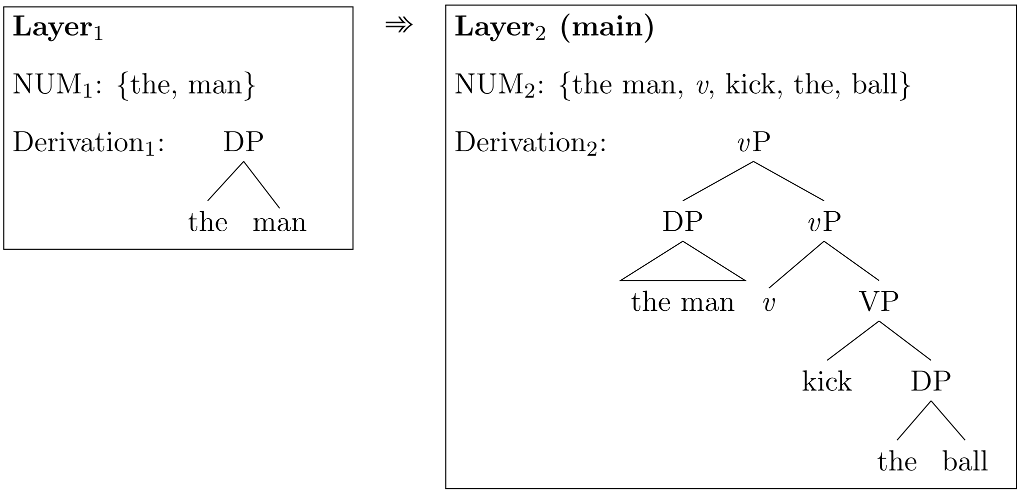
Now that we have inspected the makeup of the multidimensional coordination structure, we can put everything together and implement Form Sequence in syntax. Recall from Section 3 that Form Sequence, in Chomsky's conception, comprises three steps: conjunct set generation, sequence space formation, and sequence choosing. With the discussion in Section 4.3, we must add in a fourth step: coordinate phrase derivation. I proposed in Section 4.2 that both sequence space formation and sequence choosing (i.e., the two steps for ϵ purposes) take place in the discourse. Conjunct set generation and coordinate phrase derivation, on the other hand, must take place in the syntax, since they are directly relevant to the generation of the actual coordinate phrase.

In Section 3, I pointed out that the initial set of conjuncts required by Form

Sequence could not be formed by Set Merge due to its multimembered nature. That said, it is perfectly viable as a set of initial ingredients for the derivation of a coordinate phrase. In this sense, the role of the initial set resembles that of a Lexical Subarray in Chomsky (2000), except that Lexical Subarrays contain items selected from the lexicon, whereas the initial conjunct set contains syntactically derived conjunct phrases. The idea is that the conjuncts are prederived and "reselected" into the quasi Lexical Subarray, which then serves as the starting point of the derivation of the coordinate phrase. That is apparently also Chomsky's assumption. When analyzing the sentence in (12), Chomsky (2020) remarks that "there are two parallel things generated separately. One of them is *arrive John*; the other is *John meet Bill*."

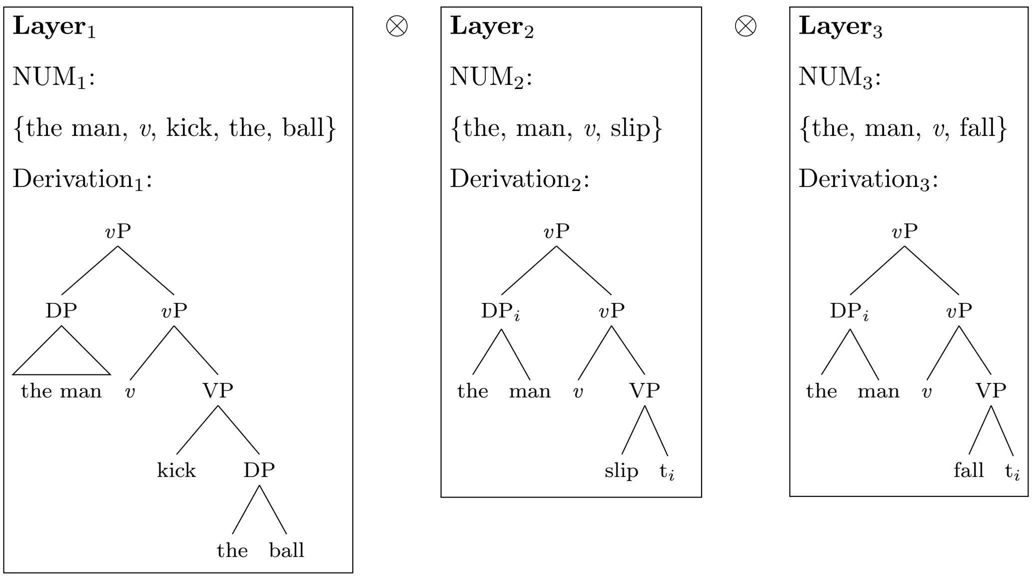
To implement this parallel derivation, I adopt Zwart's (2007, 2009, 2011) theory of Layered Derivation and assume that each conjunct is derived in a separate layer. Each derivational layer is defined by a separate numeration (NUM), which is a notion from early Minimalism (Chomsky 1995) and more or less equivalent to the later-defined Lexical Array/Subarray (Chomsky 2000). For current purposes, I do not distinguish the two terms. On Zwart's theory, complex noncomplements like subjects are constructed in separate layers before they join the main layer. Specifically, one layer's output may be included in another layer's input (i.e., its numeration). Johnson (2003) calls this mechanism renumeration. I illustrate this in (22), where ⇛ indicates a sequential relationship between derivational layers. I omit projections above vP for expository convenience.

(22) The man kicked the ball. (adapted from Zwart 2011:48)



Besides complex subjects, Zwart suggests that several other constructions can be given a layered-derivation analysis as well, including coordination. From the viewpoint of the current layer, elements derived in previous layers "have a dual nature," since they are "complex in the sense that they have been derived in a previous derivation [but] single items in that they are listed as atoms in the numeration for a subsequent derivation" (Zwart 2009:173). I illustrate the layered derivation of conjuncts in (23), where I use ⊗ to indicate a parallel relationship between derivational layers. Note that the different derivational status of the three DPs is due to Zwart's particular view on argument structure.

(23) The man kicked the ball, slipped, and fell.



I follow Chomsky in treating coordinate verbal predicates as full-fledged vPs. This calls for some careful handling of movement copies, because while each layer in (23) contains its own "copy" of *the man*, there is only one copy of it in the final sentence. In the spirit of Chomsky (2020), any of the three occurrences of *the man*—which have identical interpretations—may raise to Spec-IP, while occurrences that do not raise are indistinguishable from copies of the raised occurrence and therefore deleted across the board.

Next, the three conjuncts in (23) are renumerated into a new layer, and the new numeration additionally contains a functional category Co (from the initial, big numeration formed directly from the lexicon). This numeration is exactly the initial conjunct set Form Sequence needs. It is also at this stage that the ϵ-related process takes place in the discourse. Subsequently, the conjuncts each pair-merge with Co in a separate plane, yielding a multidimensional object as described in Section 4.3. I illustrate this in (24).

(24)

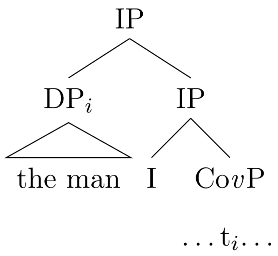
Layer$_1$ **Layer**$_2$ **Layer**$_3$	$\Rrightarrow$	**Layer**$_4$ NUM$_4$: {Co, [$_{vP_1}$ the man v_1 kick the ball]], [$_{vP_2}$ the man v_2 slip]], [$_{vP_3}$ the man v_3 fall]]} Derivation$_4$: Co vP$_1$ vP$_2$ vP$_3$

After the multidimensional CovP is derived, the next step is to merge it into the main structure, again in a new derivation layer—this time in the main layer. In the above example, the coordinate vP merges with I, and this is normal Set Merge, as in (25).

(25) **Layer**$_4$ ⇛ **Layer**$_5$ **(main)**

NUM$_5$: {CovP, I, ...}

Derivation$_5$:



One of the several occurrences of *the man* raises from vP to Spec-IP as mentioned above. Chomsky does not specify how exactly this cross-dimensional raising takes place. Here I tentatively suggest that the pivot Co, being the connection between each of the dimensions in CovP and the main dimension, may serve as a bridge or "edge" for cross-dimensional movement. This, plus the assumption that the derivation of the coordinate phrase has its own Lexical Subarray, points to the possibility that a coordination phrase is a phase in the sense of Chomsky (2001 et seq.). Finally, let us put the five derivational layers above together, which gives us the procedure below.

(26) (**Layer**$_1$ ⊗ **Layer**$_2$ ⊗ **Layer**$_3$) ⇛ **Layer**$_4$ ⇛ **Layer**$_5$

The above derivation is difficult to illustrate with conventional tree diagrams due to its multilayeredness and multidimensionality, but it can be easily illustrated by a proof tree, as in Figure 2. The proof tree also shows the complex subject layer glossed over above (i.e., that for *the man* in *the man kick the ball*). Note that the final line of the proof highly resembles Chomsky's (2020) IP structure in (12b), except that I do not treat the multidimensional CovP as a sequence in syntax but treat it as a set of pairs with a shared component (i.e., Co). This meets the definition of a partial order, more exactly one where one element is ranked above everything else. Thus, we can view multidimensional Pair Merge as an operation that takes a certain kind of numeration—one with a pivot and some syntactic objects with a common link—as input and yields a partially ordered set as output. This is exactly what happens in the step labeled "s." It is worth noting that independently of whether or not the result of "multidimensional pair merge" (pp) is really multidimensional, c-command does not obtain between the conjuncts, as they are not (contained in) sisters of each other. As mentioned in Section 4.3, the higher-dimensional talk in the literature may just be a metaphor for a non-plain-set data structure supported by natural language syntax, such as the partially ordered set proposed here.

The proof tree in Figure 2 is a comprehensive representation of the two steps conjunct set generation and coordinate phrase derivation. Now I connect these to the two other steps of Form Sequence: sequence space formation and sequence choosing. Above I have viewed a numeration of the form $\{\mathrm{Co}, \mathrm{S}^{\mathrm{L}}_1, \mathrm{S}^{\mathrm{L}}_2, \ldots, \mathrm{S}^{\mathrm{L}}_n\}$ (i.e., the initial conjunct set) as the trigger of multidimensional Pair Merge. Here I attach a further level of specialness to this numeration and propose that its generation (i.e., the renumeration) also triggers the ϵ-related process in the discourse, the result of which stays in the discourse and can be referenced by interface interpretation. Importantly, the narrow syntactic derivation and the ϵ-related discourse process do not interact with each other, as neither the renumeration step nor interface interpretation takes place in narrow syntax, and the ϵ-choice is never consulted in the course of the derivation either—recall that the product of multidimensional Pair Merge itself is not a sequence but just a partially ordered set. As such, my implementation of Form Sequence does not violate the "interpretivism tenet" in Chomskyan syntax, which says that the PF/LF interfaces merely interpret the syntactic object yielded by Narrow Syntax but do not generate further structures. A remaining question is, If the CovP in the syntactic representation is not a sequence, how can its PF linearization and LF interpretation match each other? I assume this is because both interfaces have access to the same discourse information (i.e., the ϵ-chosen sequence).

$$
\dfrac{\dfrac{}{\text{T}}\,\text{a} \qquad \dfrac{\dfrac{}{\text{Co}}\,\text{a} \quad \dfrac{\dfrac{\dfrac{}{\text{the}}\,\text{a} \quad \dfrac{}{\text{man}}\,\text{a}}{\text{the man}}\,\text{s} \quad \dfrac{\dfrac{}{v}\,\text{a} \quad \dfrac{\dfrac{}{\text{kick}}\,\text{a} \quad \dfrac{\dfrac{}{\text{the}}\,\text{a} \quad \dfrac{}{\text{ball}}\,\text{a}}{\text{the ball}}\,\text{s}}{\text{kick the ball}}\,\text{s}}{v\ \text{kick the ball}}\,\text{s}}{\text{the man } v \text{ kick the ball}}\,\text{s} \quad \dfrac{\dfrac{\dfrac{}{v}\,\text{a} \quad \dfrac{\dfrac{}{\text{slip}}\,\text{a} \quad \dfrac{\dfrac{}{\text{the}}\,\text{a} \quad \dfrac{}{\text{man}}\,\text{a}}{\text{the man}}\,\text{s}}{\text{slip the man}}\,\text{s}}{v\ \text{slip the man}}\,\text{s}}{\text{the man } v \text{ slip}}\,\text{i} \quad \dfrac{\dfrac{\dfrac{}{v}\,\text{a} \quad \dfrac{\dfrac{}{\text{fall}}\,\text{a} \quad \dfrac{\dfrac{}{\text{the}}\,\text{a} \quad \dfrac{}{\text{man}}\,\text{a}}{\text{the man}}\,\text{s}}{\text{fall the man}}\,\text{s}}{v\ \text{fall the man}}\,\text{s}}{\text{the man } v \text{ fall}}\,\text{i}}{\{\langle \text{Co}, \{\text{the man } v \text{ kick the ball}\}\rangle, \langle \text{Co}, \{\text{the man } v \text{ slip}\}\rangle, \langle \text{Co}, \{\text{the man } v \text{ fall}\}\rangle\}}\,\text{pp}}{\dfrac{\{\text{T}, \{\langle \text{Co}, \{\text{the man } v \text{ kick the ball}\}\rangle, \langle \text{Co}, \{\text{the man } v \text{ slip}\}\rangle, \langle \text{Co}, \{\text{the man } v \text{ fall}\}\rangle\}\}}{\{\text{the man}, \{\text{T}, \{\langle \text{Co}, \{\text{the man } v \text{ kick the ball}\}\rangle, \langle \text{Co}, \{\text{the man } v \text{ slip}\}\rangle, \langle \text{Co}, \{\text{the man } v \text{ fall}\}\rangle\}\}\}}\,\text{i}}\,\text{s}
$$

Figure 2: An example proof tree for multilayered, multidimensional derivation

a: axiom s: set merge

i: internal merge pp: multidimensional pair merge

4.5 PF or discourse?

In the last section, I presented an implementation of Form Sequence with two parallel processes: a partially ordered coordinate phrase is generated in syntax, and a sequence is chosen in the discourse. This means that the syntactic representation of coordination is only hierarchical but not linear (i.e., not a sequence) after all. That is nothing surprising but just the usual assumption about syntactic structures in the Minimalist Program, but it does make the ϵ-chosen sequence look somewhat like a linearization tool.

The ϵ-choice in Form Sequence indeed feels more urgently needed by PF than by LF, because unlike phonological linearization, the semantic interpretation of coordinate phrases does not strictly depend on sequential information. Truth-conditionally, conjunction is commutative and therefore not affected by operand reordering. And when coordinate phrases do get sequential readings, those tend to be pragmatic implicatures. Take the sentence in (27) for example.

(27) Yesterday I went to the post office, the supermarket, and the bookstore.

This sentence is naturally understood as saying that the speaker went to the three places in the given order. However, that reading is defeasible by a follow-up phrase *but not in that order*.

Despite the above impression that the ϵ-operator in Form Sequence is a PF-specific tool, I stick to my earlier view that it performs its function in the discourse, which I understand as the general context of speech. In the field of semantics (e.g., in Discourse Representation Theory), the discourse is mainly associated with issues like referent assignment, presupposition, and attitudes. But the discourse qua a general speech context arguably contains more linguistically relevant information. For instance, in Section 4.1 I demonstrated the relevance of discourse information for numeration formation. And as I showed in Section 4.2, common knowledge and conventionalized information are also readily available in the discourse.

Evidence for the discourse-view of the ϵ-operator comes from the observation that even though the semantic interpretation of coordination does not hinge on the ϵ-chosen sequence, the sequence and its implicature may nevertheless interfere with the normal interpretation procedure. Consider (28) for example.

(28) ?The twelve months are January, February, March, April, May, June, July, August, September, October, November, and December—but not in that order.

Unlike in (27), the addition of *but not in that order* here makes the sentence infelicitous. This is because *but not in that order* denies the implicature from the surface order of the coordination, but as that order matches the conventionalized order in this case, the phrase ends up denying the latter as well. Since the surface order of the coordination, being a result of PF linearization, strictly relies on the ϵ-chosen sequence on the Form Sequence theory, the infelicity of (28) is evidence that the ϵ-generated sequential information is not PF-specific but is generally available in the discourse, whereby it is also accessible by the semantics/pragmatics module. As further evidence, the sentence in (28) becomes felicitous if we reorder the month names, as in (29).

(29) The twelve months are March, April, January, May, June, July, February, August, October, November, September, and December—but not in that order.

This time, the sequential reading implied by the surface order (and ultimately by the ϵ-chosen order) does not match the conventional order associated with the month names, and consequently the denying of the former does not lead to infelicity. Overall, contrasts like that between (28) and (29) suggest that the ϵ-choice in Form Sequence is not as irrelevant to semantics/pragmatics as it appears. Therefore, the most suitable place to locate the sequential information is the general discourse instead of PF.

5 Form Sequence beyond syntax

5.1 An application in formal semantics

My discussion of Form Sequence so far has been limited to its original application in Chomsky's lectures. In this section, I demonstrate that Form Sequence has potential applications outside the coordination construction—and indeed outside syntax. The case I present here is that of the Konstanz School's semantic theory of (in)definite NPs and intersentential anaphora. I illustrate these phenomena in (30).

(30) a. (In)definite NPs: the man, a man, ...

b. Intersentential anaphora: A man comes. The man / He smokes.

Dissatisfied with the iota-based approach to *the*, the quantificational approach to *a*, and the E-type pronoun approach to intersentential anaphora (see, inter alia, Egli & von Heusinger 1995, von Heusinger 1997a, and Retoré 2014 for details), researchers in the Konstanz School (most representatively Klaus von Heusinger) developed a unified theory for all three phenomena above based on an extension of Hilbert's ϵ-operator. Specifically, they equipped ϵ with a context index, thus assigning a dedicated choice function to each context. Given a context c, the classical ϵ-term $\epsilon x.\,\mathrm{F}(x)$ becomes $\epsilon_c x.\,\mathrm{F}(x)$, which picks out the most salient element in $[\![\mathrm{F}]\!]$ under c. On the semantic side, ϵ_c is interpreted by an indexed choice function Φ_c. As such, there is "not *one single* choice function but *a whole family* of them indexed with situations" (Egli & von Heusinger 1995:134).

I will not go into the technical details of the Konstanz School's semantic analysis. Interested readers are referred to Egli & von Heusinger (1995) and von Heusinger (1997a,b, 2000, 2002, 2004, 2013). My focus here is just on the notion of salience their theory relies on, which is originally from Lewis (1979). With this notion, the descriptive material in a definite NP (e.g., *man* in *the man*) denotes a set as usual, but this set is furthermore equipped with a salience-based ranking of its members, which is essentially a discourse-determined sequence.

As Egli & von Heusinger (1995:134) point out, the context-indexed ϵ-operator (i.e., their "global" ϵ-operator) does two jobs at once: ranking $[\![\text{F}]\!]$ and choosing its most salient element. This is quite reminiscent of the ϵ-related process in Form Sequence, for there, too, we must first prepare the ambient set (i.e., the sequence space) and then make the choice.

The similarity between the Konstanz School's use of ϵ and the use thereof in Form Sequence goes beyond the level of the basic procedure. For instance, the ϵ-choice in both cases is semideterministic—respectively being influenced by the salience ranking mentioned above and the pragmatic constraint mentioned in Section 4.2—unlike in Hilbert's original conception, where the ϵ-operator is nondeterministic. However, the aspect of similarity I want to highlight here is not about the Konstanz School's proposed use of ϵ, but about the way their salience-based ranking is formed (which they do not specify). Since the ranking itself is a sequence and takes place in the discourse, it in principle can be generated by the ϵ-method of Form Sequence. I illustrate this possibility in (31).

(31) $\epsilon_{\langle \text{salience},\, c\rangle} X.\, \text{seq}_{[\![\text{man}]\!]^*}(X)$

In the above notation, $[\![\text{man}]\!]$ is the set to be ranked, and the tuple $\langle \text{salience}, c\rangle$ specifies the conditions that together influence the ϵ-choice. In this case, the choice is influenced by the salience parameter and the context c. As in Section 4.2, I use $[\![\text{man}]\!]^*$ to denote the free monoid on $[\![\text{man}]\!]$, which contains all sorts of sequences of men. The second-order indexed ϵ-term in (31), then, precisely picks out the sequence ordered by salience in context c.

On the above perspective, we can now reformulate the Konstanz School's analysis of definite NPs with two steps of ϵ-choice, as in (32).

(32) *the F*, context c

a. Choose sequence: $\epsilon_{\langle \text{salience},\, c\rangle} X.\, \text{seq}_{[\![\text{F}]\!]^*}(X)$

b. Choose element: $\epsilon_c x.\, [\![\mathrm{F}]\!]^{\langle \text{salience}, c\rangle}(x)$

Both steps in (32) involve a deterministic ϵ, and they furthermore "agree" in the context parameter c. Here, too, the sequence-generating procedure takes place in the discourse. Overall, the demonstration in this section suggests that Form Sequence, or at least its ϵ-based part, is potentially much more useful than originally conceived in Chomsky's lectures. In principle, whenever a sequence is needed in the discourse, it can be generated by Form Sequence.

5.2 The third-factor perspective

In the last section, I showed that the ϵ-part of Form Sequence could be applied outside syntax. Along this line of thought, in this section I make a further connection between the mechanism of Form Sequence and the "third factor" perspective in Chomsky (2005), the significance of which in the Minimalist Program is increasingly clear.

In his remarks on the biolinguistic enterprise, Chomsky (2005:6) lists "three factors that enter into the growth of language in the individual": (i) genetic endowment, (ii) experience, and (ii) principles not specific to the faculty of language. Specifically, the third factor falls in two major subtypes: (a) principles of data analysis or processing, and (b) principles of structural architecture and developmental constraints (e.g., principles of efficient computation).

The relevance of the third factor in the ϵ-method of Form Sequence is self-evident. The preparation of the sequence space (i.e., the generation of the free monoid) involves the manipulation of data structures, and the semideterministic choice of sequence is essentially a matter of decision-making. The domain-general nature of sequence construction and decision-making is uncontroversial. The former is "ubiquitous in our lives" and "important in intact cognitive processing" (Jaswal 2017:5–6), and the latter is a high-level process that "builds on more basic cognitive processes such as perception, memory, and attention" and "is uniquely identified by ... the process of *choice*" (Gonzalez 2017:249).

To illustrate the cross-domain manifestation of the ϵ-method, consider the cognitive process of prioritization, which is an important ability in real-world multitasking, especially when there is time pressure (Bai 2017). There are both multitasking scenarios that require optimal routines and scenarios that require spontaneous prioritization. These respectively correspond to sequences with conventionalized ordering (see (20)) and sequences with more arbitrary ordering (see (27)) in the linguistic domain. An example of routinized prioritization is the ABC (Airway, Breathing, Circulation) protocol in first aid, and an example of spontaneous prioritization is that in household chores. I present their "initial conjunct sets" in (33) (the chores are expository).

(33) a. $P = \{\text{Airway}, \text{Breathing}, \text{Circulation}\}$

b. $Q = \{\text{cleaning floor}, \text{washing dishes}, \text{taking out trash}, \text{cooking}, \text{feeding pets}\}$

The items in (33a) are associated with a conventional order, whereas those in (33b) are not. Therefore, the ranking of Q is more context-dependent than that of P. Accordingly, the ϵ-choice of sequence for P is also more deterministic than that for Q, though obviously the ϵ-operator is not indeterminate in the latter case either. I present the two ϵ-terms in (34).

(34) a. $\epsilon_{\langle \text{convention},\, c\rangle} X.\, \text{seq}_{[\![\text{P}]\!]^*}(X)$

b. $\epsilon_{\langle c\rangle} X.\, \text{seq}_{[\![\text{Q}]\!]^*}(X)$

As before, I use a subscript tuple (i.e., an index) to indicate the parameters influencing the ϵ-choice. The ϵ-term in (34a) chooses a sequence of first-aid steps based on both convention and the context—the latter is included because presumably there are occasions where the conventional protocol must be altered—while the ϵ-term in (34b) chooses a sequence of chores solely based on the context, which covers the hygienic state of the house, the time, the pets' level of hunger, and so on. In both cases, the ϵ-method takes place in the agent's mind, but the chosen sequences are normally turned into actions instead of language.

6 Conclusion

The ϵ-operator, also known as the choice operator, is a valuable formal tool from Hilbert's work on the foundations of mathematics. In this paper, I studied its role in Chomsky's recent Form Sequence theory for coordination, including the unbounded unstructured case thereof (Chomsky 2019, 2020). Due to the programmatic nature of Chomsky's proposal, I took it as an inspiring point of departure but developed my own version of the theory within current Minimalism.

Specifically, I argued that Form Sequence had better be viewed as a two-thread procedure, with its ϵ-related part (i.e., sequence space formation, sequence choosing) taking place in the general speech context or discourse and its coordination-related part (i.e., initial conjunct set formation, coordinate phrase derivation) taking place in syntax. For the latter, I adopted Zwart's (2007 et seq.) theory of Layered Derivation and identified the initial conjunct set as a quasi Lexical Subarray generated by renumeration. Then, I derived the coordinate phrase by the multidimensional extension of Pair Merge suggested in Chomsky (2019), with technical details filled in. I claimed that the syntactic object thus derived was still a hierarchical rather than linear structure, though it differs from set-merged objects in being partially ordered. As usual, the linear order only becomes relevant at the interfaces, where it is proffered by the ϵ-picked sequence in the speech context.

After laying out my implementation of Form Sequence, I further argued that its ϵ-part had more theoretical significance in linguistics beyond Chomsky's immediate concern (i.e., coordination). As an example, I demonstrated that the same ϵ-method can be used to generate the salience ranking (essentially a sequence) in the Konstanz School's semantic theory of (in)definite NPs, which incidentally is also ϵ-based. Interestingly, the use of ϵ in that theory and its use in Form Sequence have an important similarity—both are semideterministic, unlike in Hilbert's original conception, where the symbol is nondeterministic. Finally, I showed that the ϵ-part of Form Sequence had further manifestation beyond the domain of language (e.g., in multitasking prioritization), which makes it into a general cognitive or third-factor strategy in the sense of Chomsky (2005).

References

Abrusci, V. Michele. 2017. Hilbert's τ and ϵ in proof theoretical foundations of mathematics: An introduction. *The IfCoLog Journal of Logics and their Applications: Special Issue—Hilbert's Epsilon and Tau in Logic, Informatics, and Linguistics* 4, 257–274.

Ackermann, Wilhelm. 1925. Begründung des tertium non datur mittels der hilbertschen theorie der widerspruchsfreiheit. *Mathematische Annalen* 93(1), 1–36.

Asser, Günter. 1957. Theorie der logischen Auswahlfunktionen. *Mathematical Logic Quarterly* 3(1-5), 30–68.

Avigad, Jeremy & Richard Zach. 2020. The epsilon calculus. In Edward N. Zalta (ed.), *The Stanford encyclopedia of philosophy* (Fall 2020 edn.), Metaphysics Research Lab, Stanford University.

Bai, Hao. 2017. Cognitive processes of prioritization in multitasking. Mississippi State University dissertation.

Bernays, Paul. 1991/1958. *Axiomatic set theory.* New York: Dover Publications.

Chatzikyriakidis, Stergios, Fabio Pasquali & Christian Retoré. 2017. From logical and linguistic generics to Hilbert's tau and epsilon quantifiers. *The If-*

CoLog Journal of Logics and their Applications: Special Issue—Hilbert's Epsilon and Tau in Logic, Informatics, and Linguistics 4, 231–256.

Chomsky, Noam. 1995. *The minimalist program.* Cambridge MA: MIT Press.

Chomsky, Noam. 2000. Minimalist inquiries: The framework. In Roger Martin, David Michaels & Huan Uriagereka (eds.), *Step by step: Essays on minimalist syntax in honor of Howard Lasnik*, 89–156. Cambridge MA: MIT Press.

Chomsky, Noam. 2001. Derivation by phase. In Michael Kenstowicz (ed.), *Ken Hale: A life in language*, 1–52. Oxford: Oxford University Press.

Chomsky, Noam. 2004. Beyond explanatory adequacy. In Adriana Belletti (ed.), *Structures and beyond.* The Cartography of Syntactic Structures, vol. 3, 104–131. Oxford: Oxford University Press.

Chomsky, Noam. 2005. Three factors in language design. *Linguistic Inquiry* 36(1), 1–22.

Chomsky, Noam. 2013. Problems of projection. *Lingua* 130, 33–49.

Chomsky, Noam. 2015. Problems of projection: Extensions. In Elisa Di Domenico, Cornelia Hamann & Simona Matteini (eds.), *Structures, strategies and beyond: Studies in honour of Adriana Belletti*, 1–16. Amsterdam: John Benjamins.

Chomsky, Noam. 2019. Lecture 4, UCLA lecture series, May 2. `https://linguistics.ucla.edu/noam-chomsky/`.

Chomsky, Noam. 2020. Lecture at the 161st Meeting of the Linguistic Society of Japan, Nov 22. `https://youtu.be/X4F9NSVVVuw`.

Curry, Haskell B. 1963. *Foundations of mathematical logic.* New York: McGraw-Hill.

É. Kiss, Katalin. 2002. *The syntax of hungarian.* Cambridge: Cambridge University Press.

Egli, Urs & Klaus von Heusinger. 1995. The epsilon operator and E-type pronouns. In Urs Egli, Peter E. Pause, Christoph Schwarze, Arnim von Stechow & Götz Wienold (eds.), *Lexical knowledge in the organization of language*, 121–141. Amsterdam: John Benjamins.

Gödel, Kurt. 1931. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatshefte für Mathematik und Physik* 38(1), 173–198.

Gonzalez, Cleotilde. 2017. Decision-making: A cognitive science perspective. In Susan F. Chipman (ed.), *The Oxford handbook of cognitive science*, 249–264. Oxford: Oxford University Press.

von Heusinger, Klaus. 1997a. Definite descriptions and choice functions. In Seiki Akama (ed.), *Logic, language, and computation*, 61–91. Berlin: Springer.

von Heusinger, Klaus. 1997b. *Salienz und Referenz: Der Epsilonoperator in der Semantik der Nominalphrase und anaphorischer Pronomen.* Berlin: Akademie Verlag GmbH.

von Heusinger, Klaus. 2000. The reference of indefinites. In Klaus von Heusinger & Urs Egli (eds.), *Reference and anaphoric relations*, 247–265. Dordrecht: Springer.

von Heusinger, Klaus. 2002. Specificity and definiteness in sentence and discourse structure. *Journal of Semantics* 19(3), 245–274.

von Heusinger, Klaus. 2004. Choice functions and the anaphoric semantics of definite NPs. *Research on Language and Computation* 2(3), 309–329.

von Heusinger, Klaus. 2013. The salience theory of definiteness. In Alessandro Capone, Franco Lo Piparo & Marco Carapezza (eds.), *Perspectives on linguistic pragmatics*, 349–374. Cham: Springer.

Hilbert, David & Paul Bernays. 1939. *Grundlagen der Mathematik*, vol. 2. Berlin: Springer.

Jaswal, Snehlata. 2017. Editorial: What next—The cognition of sequences. *Frontiers in Psychology* 8. doi:10.3389/fpsyg.2017.02160.

Johnson, Kyle. 2003. Towards an etiology of adjunct islands. *Nordlyd* 31(1), 187–215.

Leisenring, Albert C. 1969. *Mathematical logic and Hilbert's ϵ symbol.* London: Macdonald & Co. Ltd.

Leiß, Hans. 2017. On equality of contexts and completeness of the indexed ϵ-calculus. *The IfCoLog Journal of Logics and their Applications: Special Issue—Hilbert's Epsilon and Tau in Logic, Informatics, and Linguistics* 4, 347–366.

Lewis, David. 1979. Scorekeeping in a language game. In Rainer Bäuerle, Urs Egli & Arnim von Stechow (eds.), *Semantics from different points of view*, 172–187. Berlin: Springer.

Mints, G.E. & Darko Sarenac. 2003. Completeness of indexed ϵ-calculus. *Archive for Mathematical Logic* 42(7), 617–625.

Retoré, Christian. 2014. Typed Hilbert epsilon operators and the semantics of determiner phrases. In Glyn Morrill, Reinhard Muskens, Rainer Osswald & Frank Richter (eds.), *Proceedings for the 19th international conference on formal grammar*, 15–33. Heidelberg: Springer.

Slater, Hartley. 2017. $(\exists y)(y = \epsilon x f x)$. *The IfCoLog Journal of Logics and their Applications: Special Issue—Hilbert's Epsilon and Tau in Logic, Informatics, and Linguistics* 4, 276–286.

de Vries, Mark. 2004. Parataxis as a different type of asymmetric Merge. In *Proceedings of the Interfaces Conference hosted by the International Conference on Advances in the Internet, Processing, Systems, and Interdisciplinary research (CD-ROM)*, .

de Vries, Mark. 2005. Coordination and syntactic hierarchy. *Studia Linguistica* 59, 83–105.

Whitehead, Alfred N. & Bertrand Russell. 1910. *Principia mathematica*, vol. 1. Cambridge: Cambridge University Press.

Zwart, Jan-Wouter. 2007. Layered derivations. Advanced Core Training in Linguistics lecture, Mar 9, London.

Zwart, Jan-Wouter. 2009. Prospects for top-down derivation. *Catalan Journal of Linguistics* 8, 161–187.

Zwart, Jan-Wouter. 2011. Recursion in language: A layered-derivation approach. *Biolinguistics* 5(1-2), 043–056.